Electromagnetic energy for a charged Kerr black hole in a uniform magnetic field

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With the Komar mass formula we calculate the electromagnetic energy for a charged Kerr black hole in a uniform magnetic field. We find that the total electromagnetic energy takes the minimum when the Kerr black hole possesses a nonzero net charge $Q=2\xi B_0 J_H$ where B_0 is the strength of the magnetic field, J_H is the angular momentum of the black hole, and ξ is a dimensionless parameter determined by the spin of the black hole. However, the Wald state with $Q=2B_0J_H$ does not minimize the electromagnetic energy.

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Whether an astrophysical black hole can possess a net electric charge is an interesting and important question. For an astrophysical black hole without a magnetic field the answer seems to be clear: usually the black hole cannot possess much net electric charges since otherwise the black hole will selectively accrete particles with an opposite sign of charge from the ambient material and be quickly neutralized [1,2]. (However, if accretion onto a black hole produces a luminosity close to the Eddington limit, the black hole can acquire a net charge due to the different radiation drag on electrons and ions [3].) But for a black hole immersed in a magnetic field—which is believed to occur in many astrophysical systems—the situation is different and the answer is quite unclear. Wald has shown that when a Kerr black hole is immersed in a uniform magnetic field aligned with its rotation axis, the hole acquires a net electric charge $Q_W=2B_0J_H$, where B_0 is the strength of the magnetic field and J_H is the angular momentum of the black hole (throughout the paper we use the geometric units $G=c=1$) [4]. Wald derived his result from the requirement that the ''injection energy'' along the symmetry axis should be zero for the equilibrium state. However, it can be shown that the ''injection energy'' defined by Wald in [4] depends on the path along which the injection is made and the effect of off-axis accretion of charges is unclear. So it remains a question, what is the equilibrium state for a Kerr black hole in a uniform magnetic field and whether the equilibrium state acquires a net charge.

Ruffini and Treves have analyzed the problem of a magnetized rotating sphere in flat spacetime [5]. They have found that the sphere acquires a nonzero net charge in order to minimize the total electromagnetic energy of the system. This is similar to the conclusion of Wald but the acquired charge has the opposite sign. The method of extremizing the total energy is better than that of ''injection energy'' since the former is a global approach which does not depend on the details of injection path and accretion process. Therefore in this paper we try to calculate the total electromagnetic energy for a charged Kerr black hole in a uniform magnetic field. Energy (mass) is well defined for a stationary system which is asymptotically flat and has an asymptotic timelike Killing vector [2]. However, if the uniform magnetic field extends to infinity in space, the total electromagnetic energy diverges. To obtain a finite electromagnetic energy we truncate the electromagnetic field with a spherical surface. The black hole sits at the center of the sphere, the radius of the sphere is much larger than the radius of the black hole horizon. Inside the sphere the magnetic field is uniform but outside the sphere the magnetic field decreases quickly with increasing radius. (Currents are induced on the surface of the sphere to connect the magnetic fields inside and outside the sphere.) We will calculate the total electromagnetic energy inside the sphere using the Komar mass formula [6] and show that when the total electromagnetic energy takes the minimum the black hole acquires a nonzero net charge $Q=2\xi B_0J_H=\xi Q_W$, where the dimensionless parameter ξ is a function of a/M_H and $0\leqslant\xi\leqslant[\frac{3}{2}(2+\pi)]^{-1}\approx0.13$ for $0\leqslant a/M_H\leqslant1$, M_H is the mass of the black hole and $a=J_H/M_H$. (The value of ξ is independent of the radius of the truncation sphere in the limit that the radius of the sphere is much larger than the radius of the black hole.) So the Wald state (which corresponds to $\xi=1$) does not minimize the total electromagnetic energy. Indeed, the energy of the Wald state is higher than that of the state with no charge.

For a charged rotating black hole immersed in a uniform magnetic field with the electromagnetic field being sufficiently weak ($Q^2/M_H^2\ll1$ and $B_0^2M_H^2\ll1$), the spacetime can be described with the Kerr metric. (In other words, the electromagnetic field is treated as test field in the background of Kerr spacetime.) In Boyer-Lindquist coordinates, the Kerr metric is

$$ds^2=-\left(1-\frac{2M_Hr}{\Sigma}\right)dt^2-\frac{4M_Har}{\Sigma}\sin^2\theta\;\,dt\,d\phi+\frac{\Sigma}{\Delta}dr^2+\Sigma\,d\theta^2+\frac{C\sin^2\theta}{\Sigma}d\phi^2, \tag{1}$$

where M_H is the Komar mass of the black hole, a is the specific angular momentum of the black hole (the angular momentum of the black hole is $J_H=M_Ha$), and

$$\Delta=r^2-2M_Hr+a^2,\quad \Sigma=r^2+a^2\cos^2\theta,$$
$$C=(r^2+a^2)^2-\Delta a^2\sin^2\theta. \tag{2}$$

The electromagnetic vector potential is [4]

$$A^a=\frac{B_0}{2}\left[\left(\frac{\partial}{\partial\phi}\right)^a+2a\left(\frac{\partial}{\partial t}\right)^a\right]-\frac{Q}{2M_H}\left(\frac{\partial}{\partial t}\right)^a, \tag{3}$$

where B_0 is the strength of the external magnetic field and Q is the electric charge of the black hole. The electromagnetic field F_{ab} is given by

$$F_{ab}=\nabla_a A_b-\nabla_b A_a. \tag{4}$$

The tensor indexes are raised and lowered with the Kerr metric g_{ab}.

Since $(\partial/\partial t)^a$ and $(\partial/\partial\phi)^a$ are the Killing vectors of Kerr spacetime and the Ricci tensor $R_{ab}=0$ for the Kerr metric, F_{ab} solves vacuum Maxwell's equations $\nabla_a F^{ab}=0$ [4]. The stress-energy tensor of the electromagnetic field is

$$T_{ab}=\frac{1}{4\pi}\left(F_{ac}F_b{}^c-\frac{1}{4}g_{ab}F_{de}F^{de}\right). \tag{5}$$

The trace of the stress-energy tensor of the electromagnetic field is $T=g_{ab}T^{ab}=0$.

With the Komar mass formula [6], the total mass (energy) for the Kerr black hole and the electromagnetic field (with suitable truncation as described earlier) is [2,7–9]

$$M=2\int_\Sigma\left(T_{ab}-\frac{1}{2}Tg_{ab}\right)n^a\left(\frac{\partial}{\partial t}\right)^b dV+\frac{1}{4\pi}\kappa\mathcal{A}+2\Omega_H J_H, \tag{6}$$

where

$$n^a=\frac{1}{\alpha}\left[\left(\frac{\partial}{\partial t}\right)^a+\frac{2aM_H r}{C}\left(\frac{\partial}{\partial\phi}\right)^a\right],\quad \alpha\equiv\left(\frac{\Delta\Sigma}{C}\right)^{1/2}, \tag{7}$$

is the unit future-pointing normal to the hypersurface $\Sigma(t=\text{constant})$, dV is the volume element on Σ, $\mathcal{A}=4\pi(r_H^2+a^2)$ is the area of the horizon, $r_H=M_H+\sqrt{M_H^2-a^2}$ is the radius of the horizon, $\kappa=(r_H-M_H)/(2M_H r_H)$ is the surface gravity of the black hole, and $\Omega_H=a/(2M_H r_H)$ is the angular velocity of the black hole. So the total energy of the electromagnetic field within the truncation sphere is

$$\mathcal{E}_{EM}=\int_{r_H}^R dr\int_0^\pi d\theta\int_0^{2\pi}d\phi\ \ \epsilon\sqrt{h},$$

$$\epsilon\equiv 2\left(T_{ab}-\frac{1}{2}Tg_{ab}\right)n^a\left(\frac{\partial}{\partial t}\right)^b, \tag{8}$$

where

$$\sqrt{h}=\left(\frac{\Sigma C}{\Delta}\right)^{1/2}\sin\theta \tag{9}$$

is the measure of volume on Σ. The integration over r is truncated at $r=R$ (the radius of the truncation sphere), otherwise the integration diverges since the magnetic field is asymptotically uniform. Suppose R is $\gg r_H$ but $\ll(M_H/B_0^2)^{1/3}$ [so that $\mathcal{E}_{EM}\ll M_H$ and the background spacetime can be well approximated with the Kerr geometry]. The energy so defined is conserved since the spacetime is stationary.

Inserting Eqs. (3)–(5) into Eq. (8), we obtain

$$\epsilon=\frac{\alpha}{4\pi\Sigma^3}(f_0B_0^2+f_1Q^2+f_2B_0Q), \tag{10}$$

where

$$f_0=r^5(r-2M_H\sin^2\theta)+a^2r^2[3r^2\cos^2\theta+2M_Hr(1+\cos^2\theta)(1-3\cos^2\theta)-M_H^2(1+\cos^2\theta)(3-5\cos^2\theta)]$$
$$+a^4[3r^2\cos^4\theta-2M_Hr\cos^2\theta(3+\cos^2\theta)+M_H^2(1+\cos^2\theta)^2(2-\cos^2\theta)]+a^6\cos^6\theta, \tag{11}$$

$$f_1=r^2+a^2(2-\cos^2\theta), \tag{12}$$

$$f_2=-2a\{r^2(r-M_H)(1-3\cos^2\theta)-a^2[r\cos^2\theta(3-\cos^2\theta)-M_H(1+\cos^2\theta)(2-\cos^2\theta)]\}, \tag{13}$$

and

$$\mathcal{E}_{EM}=\mathcal{E}_0+\left(\frac{Q^2}{M_H}\right)F_1+\left(\frac{B_0J_HQ}{M_H}\right)F_2, \tag{14}$$

where (in the limit $R\gg r_H$)

$$\mathcal{E}_0=\frac{1}{3}B_0^2R^3+\mathcal{O}(R^2), \tag{15}$$

$$F_1=F_1(s)=\frac{1}{2s^3}(1-\sqrt{1-s^2})\left[s+2\arctan\left(\frac{s}{1+\sqrt{1-s^2}}\right)\right]+\mathcal{O}\left(\frac{1}{R}\right), \tag{16}$$

$$F_2=F_2(s)=\frac{2}{3s^3}\left[s(3\sqrt{1-s^2}-s^2)-6(1-s^2)\times\arctan\left(\frac{s}{1+\sqrt{1-s^2}}\right)\right]+\mathcal{O}\left(\frac{1}{R^2}\right), \tag{17}$$

where $s \equiv a/M_H$ is the spin parameter of the black hole. Since $0 \leqslant s \leqslant 1$, $F_1 > 0$ and $F_2 \leqslant 0$ always. Since the total electromagnetic field is the superposition of the external magnetic field B_0 and the electromagnetic field generated by the charge Q, the total electromagnetic energy is composed of three parts: (1) the ''bare'' energy $\mathcal{E}_0$, i.e., the energy of the external magnetic field, which is proportional to B_0^2; (2) the energy of the electromagnetic field generated by the charge Q, which is proportional to Q^2 (cf. [10] for the electromagnetic energy of a Kerr-Newman geometry); (3) the energy arising from the interaction between the external magnetic field B_0 and the electromagnetic field of the charge Q, which is proportional to $B_0 Q$. Though as $R \to \infty$ the ''bare'' energy $\mathcal{E}_0$ diverges, F_1 and F_2 converge. So F_1 and F_2 do not depend on where the truncation is if $R \gg r_H$. In particular, the difference in the electromagnetic energy between a charged Kerr black hole in a uniform magnetic field and its uncharged state, $\Delta\mathcal{E} = \mathcal{E}_{EM} - \mathcal{E}_0$, is independent of the truncation and so is well-defined.

$\mathcal{E}_{EM}$ has a minimum since $F_1(s) > 0$ for $0 \leqslant s \leqslant 1$. By $\partial\mathcal{E}_{EM}/\partial Q = 0$ we obtain

$$Q = 2\xi(s) B_0 J_H, \tag{18}$$

where

$$\xi(s) = -\frac{F_2}{4F_1} = \frac{s^3 - 3s\sqrt{1-s^2} + 6(1-s^2)\arctan\left(\frac{s}{1+\sqrt{1-s^2}}\right)}{3(1-\sqrt{1-s^2})\left[s + 2\arctan\left(\frac{s}{1+\sqrt{1-s^2}}\right)\right]}. \tag{19}$$

For $0 \leqslant s \leqslant 1$ we have $0 \leqslant \xi \leqslant [\frac{3}{2}(2+\pi)]^{-1} \approx 0.13$. $\xi(s)$ is plotted in Fig. 1. ξ decreases quickly as s decreases. As examples: $\xi(0) = 0$, $\xi(0.1) \approx 3.4 \times 10^{-4}$, $\xi(0.5) \approx 9.9 \times 10^{-3}$, $\xi(0.9) \approx 5.6 \times 10^{-2}$, $\xi(0.99) \approx 0.10$, and $\xi(1) \approx 0.13$. So, a Kerr black hole immersed in a uniform magnetic field acquires a nonzero net charge [given by Eq. (18)] to attain the minimum electromagnetic energy. Relative to the bare state (i.e., the state with $Q = 0$), the value of the minimum electromagnetic energy is

$$\Delta\mathcal{E}_{min} = -\left(\frac{F_2^2}{4F_1}\right)\frac{B_0^2 J_H^2}{M_H}. \tag{20}$$

For a Kerr black hole with $a/M_H = 0.99$ we have $Q \approx 0.20 B_0 J_H$ and $\Delta\mathcal{E}_{min} \approx -0.045 B_0^2 J_H^2/M_H$.

Wald's result $Q_W = 2B_0 J_H$ does not correspond to the lowest energy state of the electromagnetic field. In fact, the difference in the electromagnetic energy between the Wald state and the uncharged state is

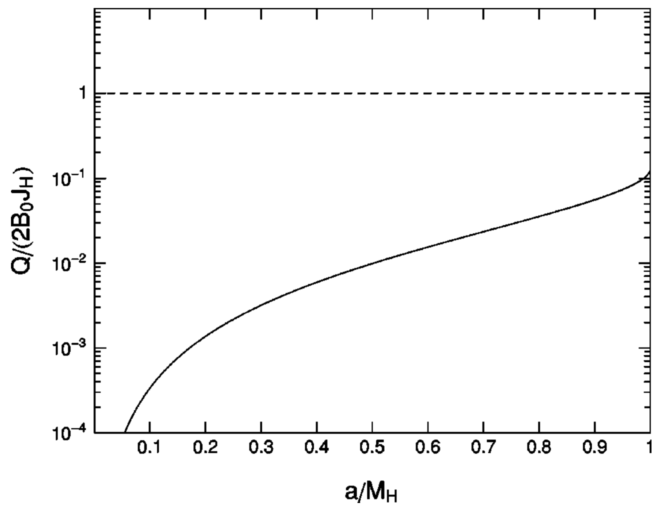

FIG. 1. A Kerr black hole immersed in a uniform magnetic field acquires a net charge $Q = 2\xi B_0 J_H$ when the total electromagnetic energy takes the minimum. The dimensionless parameter ξ is a function of $s = a/M_H$, which is shown with the solid curve. For reference the Wald charge ($\xi = 1$) is also shown with the dashed line, which does not correspond to the minimum energy of the electromagnetic field.

$$\Delta\mathcal{E}_W = 2(2F_1 + F_2)\frac{B_0^2 J_H^2}{M_H}, \tag{21}$$

which is always positive for $0 \leqslant a/M_H \leqslant 1$. (For a Kerr black hole with $a/M_H = 0.99$ we have $\Delta\mathcal{E}_W = 3.4 B_0^2 J_H^2/M_H$.) So the electromagnetic energy for the Wald state is even higher than that of the uncharged state.

To see if the charge given by Eq. (18) is important in practice, let us compare the contribution of the charge Q and the magnetic field B_0 to the magnetic flux through the northern hemisphere of the black hole horizon. The magnetic flux contributed by Q is $\Phi_Q = 2\pi a Q/r_H$. The magnetic flux contributed by B_0 is $\Phi_{B_0} = 2\pi B_0 M_H r_H (1 - a^2/r_H^2)$. (The total magnetic flux is $\Phi = \Phi_Q + \Phi_{B_0}$.) For Q given by Eq. (18) we have

$$\frac{\Phi_Q}{\Phi_{B_0}} = \frac{\xi s^2}{1 - s^2 + \sqrt{1-s^2}}, \tag{22}$$

which increases with increasing s. We find that $0 \leqslant \Phi_Q/\Phi_{B_0} < 1$ if $0 \leqslant s < 0.995$, $1 < \Phi_Q/\Phi_{B_0} < \infty$ if $0.995 < s \leqslant 1$. So the charge given by Eq. (18) is important only if $a/M > 0.995$.

In conclusion, we have calculated the electromagnetic energy for a charged Kerr black hole immersed in a uniform magnetic field. We have found that a nonzero net charge is acquired by the Kerr black hole for attaining the minimum electromagnetic energy. The Wald state is not the state with minimum electromagnetic energy. Indeed the electromagnetic energy for the Wald state is even higher than that for the uncharged state. Though the realistic case for an astrophysical black hole is much more complicated than the simple model investigated here, due to the appearance of many charged particles in the neighborhood of the black hole, our results have shown that a Kerr black hole in the lowest energy state in an external magnetic field acquires a nonzero net charge.

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